

On Type-I Quantum Affine Superalgebras

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Abstract:

The type-I simple Lie-superalgebras are $sl(m|n)$ and $osp(2|2n)$. We study the quantum deformations of their untwisted affine extensions $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$.

We identify additional relations between the simple generators (“extra q -Serre relations”) which need to be imposed to properly define $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$.

We present a general technique for deriving the spectral parameter dependent R-matrices from quantum affine superalgebras.

We determine the R-matrices for the type-I affine superalgebra $U_q(sl(m|n)^{(1)})$ in various representations, thereby deriving new solutions of the spectral-dependent Yang-Baxter equation. In particular, because this algebra possesses one-parameter families of finite-dimensional irreps, we are able to construct R-matrices depending on two additional spectral-like parameters, providing generalizations of the free-fermion model.

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1 Introduction

Quantum affine algebras are q -deformations [13, 14, 18, 19, 20] of the enveloping algebras of affine Kac-Moody algebras [23]. They were introduced as a powerful tool for the construction of solutions of the spectral parameter dependent Yang-Baxter equation. They are therefore the algebraic structures underlying the integrable models of statistical mechanics (commuting transfer matrices) as well as the quantum integrable field theories (quantum inverse scattering method, factorizable scattering matrices).

There are however solutions to the Yang-Baxter equation, and thus there are integrable models, which do not come from a quantum affine bosonic algebra. Examples are the Perk-Schultz model [8, 30] and the free fermion model [2]. It is now understood that the algebras underlying these models are quantum affine superalgebras. In spite of their significance, quantum affine superalgebras have so far remained essentially unstudied in the literature. In this paper we will study the type-I untwisted affine superalgebras $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$.

Lie superalgebras are by no means just straightforward generalizations of their bosonic counterparts. Rather they are much richer structures and have a more complicated representation theory [21, 22]. Two interesting effects occur when one defines the quantum deformations of the enveloping algebras of Lie superalgebras. One is that a fixed Lie superalgebra allows many inequivalent systems of simple roots and that these give rise to different Hopf algebras upon deformation. They are related by twistings [26]. The other is that the simple raising and lowering generators of a Lie superalgebra obey more relations than just the usual Serre relations known from bosonic Lie algebras [34, 35, 24, 25, 36, 37]. We give the necessary extra Serre relations to properly define $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$ in section 2.

To every pair of finite-dimensional irreps of a quantum affine algebra there exists a spectral parameter dependent R-matrix. It can in principle be constructed by inserting the representation matrices into the general formula for the universal R-matrix. The universal R-matrices are known also for quantum affine superalgebras [25]. In practice however, this method is too difficult and therefore, in section 3, we present a practical method to obtain the R-matrices. Our method immediately gives the spectral decomposition of the R-matrices, i.e., it gives the R-matrices in the form

$$\check{R}(x) = \sum_{\nu} \rho_{\nu}(x) \mathbf{P}_{\nu} \quad (1.1)$$

where the \mathbf{P}_{ν} are elementary intertwining operators of the non-affine algebra and the $\rho_{\nu}(x)$ are meromorphic functions of the spectral parameter x . In section 4 we determine these functions $\rho_{\nu}(x)$ explicitly for a large number of examples.

Type-I superalgebras are particularly interesting because they possess one-parameter families of finite-dimensional irreps. The R-matrices for a pair of such representations will then also depend on these extra parameters. These parameters enter the Yang-Baxter equation in a

similar way as the spectral parameter, though in a non-additive form. An R-matrix of this form, depending on three “spectral” parameters, has been known: the R-matrix of the free fermion model [2]. This paper gives the quantum group theoretic interpretation for this R-matrix (in its trigonometric form): it is the R-matrix for the tensor product of two parameter-dependent 2-dimensional irreps of $U_q(sl(1|1)^{(1)})$. This allows us to give the generalizations of this R-matrix for other type-I affine superalgebras in sections 4.2. We expect that these R-matrices will find interesting physical applications.

This paper is organized as follows: In section 2 we give the definition of the type-I untwisted quantum affine superalgebras $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$. In particular we identify extra Serre relations needed to define them. In section 3 we give a general technique for deriving the spectral parameter dependent R-matrices from these quantum affine superalgebras. In section 4 we apply this technique to derive a large number of new R-matrices and in particular the novel R-matrices depending on extra spectral-like parameters. Section 5 contains a discussion of our results. For the convenience of the reader, two appendices review some known material on the representation theory of superalgebras. Appendix A reviews Kac’s induced module construction and appendix B gives the results about the classification of finite-dimensional unitary irreps of type-I superalgebras.

2 Definition of Quantum Affine Superalgebras and extra Serre relations

In this section we want to define a quantum affine superalgebra $U_q(\mathcal{G}^{(1)})$ as the quantum deformation, depending on a nonzero parameter $q \in \mathbf{C}$, of the universal enveloping algebra of an untwisted simple affine superalgebra $\mathcal{G}^{(1)}$.

Let \mathcal{G} be a simple superalgebra [21]. Throughout the paper we choose the distinguished[‡] set of simple roots α_i , $i = 1, \dots, r$. Let $(\ , \)$ be a fixed invariant bilinear form on the root space of \mathcal{G} . Let $\mathcal{G}^{(1)}$ denote the untwisted affine superalgebra associated with \mathcal{G} . It possesses an additional simple root $\alpha_0 = \delta - \psi$, where δ is the minimal imaginary root of $\mathcal{G}^{(1)}$ satisfying $(\delta, \delta) = 0 = (\delta, \alpha_i)$, $\forall i$, and ψ denotes the highest root of \mathcal{G} . The generalized Cartan matrix $A = (a_{ij})_{0 \leq i, j \leq r}$ is defined from the simple roots by

$$a_{ij} = \begin{cases} \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, & \text{if } (\alpha_i, \alpha_i) \neq 0 \\ (\alpha_i, \alpha_j), & \text{if } (\alpha_i, \alpha_i) = 0 \end{cases} \quad (2.1)$$

In order to be able to define the quantum deformation $U_q(\mathcal{G}^{(1)})$ we present $\mathcal{G}^{(1)}$ entirely in terms of its generators $\{e_i, f_i, h_i, i = 0, 1, \dots, r\}$ corresponding to the simple roots. In analogy

[‡] As mentioned in the introduction, (affine) superalgebras allow many inequivalent systems of simple roots. See [21]. The relation between the different quantum superalgebras obtained by choosing different systems of simple roots is studied in [26].

to the procedure used for bosonic Lie algebras we define $\overline{\mathcal{G}^{(1)}}$ as the Lie superalgebra generated by the simple generators $\{e_i, f_i, h_i, i = 0, 1, \dots, r\}$ [§] subject to the relations

$$\begin{aligned} [h_i, h_j] &= 0, & [e_i, f_j] &= \delta_{ij} h_j, \\ [h_i, e_j] &= (\alpha_i, \alpha_j) e_j, & [h_i, f_j] &= -(\alpha_i, \alpha_j) f_j, \\ [e_i, e_i] &= [f_i, f_i] = 0, & \text{if } (\alpha_i, \alpha_i) &= 0, \\ (ad e_i)^{1-a_{ij}} e_j &= 0, & (ad f_i)^{1-a_{ij}} f_j &= 0, \quad \text{if } (\alpha_i, \alpha_i) \neq 0, \quad i \neq j. \end{aligned} \quad (2.2)$$

Here $[,]$ denotes the Lie superbracket which satisfies $[a, b] = -(-1)^{[a][b]}[b, a]$ where $[a] \in \mathbf{Z}_2$ denotes the degree of the element a , and $(ad a) b \equiv [a, b]$. The last relations in eq. (2.2) are the usual Serre relations.

The universal enveloping algebra $\overline{U(\mathcal{G}^{(1)})}$ of $\overline{\mathcal{G}^{(1)}}$ is the unital \mathbf{Z}_2 -graded associative algebra generated by the same generators and the same relations eq. (2.2) where now however $[,]$ denotes the (anti)commutator $[a, b] = ab - (-1)^{[a][b]}ba$. The Lie superalgebra $\overline{\mathcal{G}^{(1)}}$ is naturally embedded in $\overline{U(\mathcal{G}^{(1)})}$.

While in the purely bosonic case $\overline{\mathcal{G}^{(1)}} = \mathcal{G}^{(1)}$, this is not generally true for Lie superalgebras. Rather, the Lie superalgebra $\overline{\mathcal{G}^{(1)}}$ defined above generally contains a proper maximal Lie algebra ideal M and the affine superalgebra $\mathcal{G}^{(1)}$ is obtained as the quotient $\overline{\mathcal{G}^{(1)}}/M$ [34, 35]. The extra relations among the simple generators which hold in $\mathcal{G}^{(1)}$ are referred to as “extra Serre relations”. The universal enveloping algebra $U(\mathcal{G}^{(1)})$ is obtained as the quotient $U(\mathcal{G}^{(1)}) = \overline{U(\mathcal{G}^{(1)})}/\tilde{M}$ where \tilde{M} is the two-sided ideal in $\overline{U(\mathcal{G}^{(1)})}$ generated by M .

We now introduce the quantum deformation $\overline{U_q(\mathcal{G}^{(1)})}$ of the universal enveloping algebra $\overline{U(\mathcal{G}^{(1)})}$ as the unital \mathbf{Z}_2 -graded associative algebra generated by $\{e_i, f_i, q^{\pm h_i}, i = 0, 1, \dots, r\}$ subject to the relations

$$\begin{aligned} q^h \cdot q^{h'} &= q^{h+h'}, & h, h' &= \pm h_i, \quad i = 0, 1, \dots, r \\ q^{h_i} e_j q^{-h_i} &= q_i^{a_{ij}} e_j, & q^{h_i} f_j q^{-h_i} &= q_i^{-a_{ij}} f_j \\ [e_i, f_j] &= \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \\ e_i^2 &= f_i^2 = 0, & \text{if } (\alpha_i, \alpha_i) &= 0 \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k &= 0, & (i \neq j), & (\alpha_i, \alpha_i) \neq 0 \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k &= 0, & (i \neq j), & (\alpha_i, \alpha_i) \neq 0 \end{aligned} \quad (2.3)$$

[§]Note that throughout this paper we define the affine algebras without the derivation d . It can be reintroduced at any stage without any complications.

where

$$\left[\begin{matrix} 1 - a_{ij} \\ k \end{matrix} \right]_{q_i} = \begin{cases} \left[\begin{matrix} 1 - a_{ij} \\ k \end{matrix} \right]_{q_i}^-, & \text{if } \alpha_i \text{ is even} \\ (-)^{k(k-(-1)^{[\alpha_j]})/2} \left[\begin{matrix} 1 - a_{ij} \\ k \end{matrix} \right]_{q_i}^+, & \text{if } \alpha_i \text{ is odd} \end{cases} \quad (2.4)$$

with

$$\left[\begin{matrix} m \\ n \end{matrix} \right]_q^\pm = \begin{cases} \frac{[m]^\pm!}{[m-n]^\pm![n]^\pm!}, & m \geq n \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad (2.5)$$

$$[m]^\pm! = [m]^\pm [m-1]^\pm \cdots [0]^\pm, \quad [m]^\pm = \frac{q^m \pm q^{-m}}{q - q^{-1}}, \quad [0]^\pm = 1$$

$$q_i = \begin{cases} q^{(\alpha_i, \alpha_i)/2}, & \text{if } (\alpha_i, \alpha_i) \neq 0 \\ q, & \text{if } (\alpha_i, \alpha_i) = 0 \end{cases} \quad (2.6)$$

Throughout this paper we will assume that q is generic, i.e. not a root of unity. In the limit $q \rightarrow 1$ the above relations reproduce the relations eq. (2.2) and thus $\overline{U_q(\mathcal{G}^{(1)})}$ goes over to $\overline{U(\mathcal{G}^{(1)})}$.

The algebra $\overline{U_q(\mathcal{G}^{(1)})}$ is a Hopf algebra. The coproduct is given by

$$\begin{aligned} \Delta(q^{\pm h}) &= q^{\pm h} \otimes q^{\pm h} \\ \Delta(e_i) &= e_i \otimes q^{-h_i/2} + q^{h_i/2} \otimes e_i \\ \Delta(f_i) &= f_i \otimes q^{-h_i/2} + q^{h_i/2} \otimes f_i \end{aligned} \quad (2.7)$$

We omit the formulas for the antipode and the counit.

Finally we are ready to define the quantum affine superalgebra $U_q(\mathcal{G}^{(1)})$ as the quotient $U_q(\mathcal{G}^{(1)}) = \overline{U_q(\mathcal{G}^{(1)})} / \tilde{M}_q$, where \tilde{M}_q is a proper two-sided Hopf algebra ideal in $\overline{U_q(\mathcal{G}^{(1)})}$ which, in the limit $q \rightarrow 1$, goes over to the ideal \tilde{M} in $\overline{U(\mathcal{G}^{(1)})}$. This ensures that $U_q(\mathcal{G}^{(1)})$ is a Hopf algebra which goes over to the undeformed algebra $U(\mathcal{G}^{(1)})$ in the limit of $q \rightarrow 1$. The extra relations among the simple generators which arise from the division by the ideal \tilde{M}_q are referred to as “extra q -Serre relations”. In the following we will identify the extra q -Serre relations for $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$.

All of the above definitions can also be taken over to the non-affine case with the only change that all indices run only from 1 to r , i.e., there are no e_0, f_0 and h_0 .

2.1 Extra q -Serre relations for $U_q(sl(m|n)^{(1)})$

Suitable extra q -Serre relations for the case $\mathcal{G} = sl(m|n)$ have been derived in [34, 35], see also note added to [24]. These can be straightforwardly extended to $\mathcal{G}^{(1)} = sl(m|n)^{(1)}$.

For $sl(m|n)^{(1)}$ we take the set of simple roots,

$$\alpha_0 = \delta - \epsilon_1 + \delta_n,$$

$$\begin{aligned}
\alpha_i &= \epsilon_i - \epsilon_{i+1}, & i &= 1, 2, \dots, m-1 \\
\alpha_m &= \epsilon_m - \delta_1, \\
\alpha_{m+j} &= \delta_j - \delta_{j+1}, & j &= 1, 2, \dots, n-1
\end{aligned} \tag{2.8}$$

with δ , $\{\epsilon_i\}_{i=1}^m$ and $\{\delta_i\}_{i=1}^n$ satisfying

$$\begin{aligned}
(\delta, \delta) &= (\delta, \epsilon_i) = (\delta, \delta_i) = 0, & (\epsilon_i, \epsilon_j) &= \delta_{ij}, \\
(\delta_i, \delta_j) &= -\delta_{ij}, & (\epsilon_i, \delta_j) &= 0
\end{aligned} \tag{2.9}$$

The associated Dynkin diagram is

$$\begin{array}{c}
\alpha_0 \\
\downarrow \\
1 \\
\swarrow \quad \searrow \\
\begin{array}{ccccccc}
1 & 1 & & 1 & 1 & 1 & \\
\circ & \circ & & \circ & \otimes & \circ & \\
\alpha_1 & \alpha_2 & & \alpha_{m-1} & \alpha_m & \alpha_{m+1} & \\
& & & & & & \alpha_{m+n-2} \quad \alpha_{m+n-1}
\end{array}
\end{array} \tag{2.10}$$

where the grey nodes \otimes correspond to the fermionic simple roots.

In addition to the extra q -Serre relations given in [34, 35] for $U_q(sl(m|n))$,

$$\begin{aligned}
&e_m e_{m-1} e_m e_{m+1} + e_{m-1} e_m e_{m+1} e_m + e_m e_{m+1} e_m e_{m-1} + \\
&\quad + e_{m+1} e_m e_{m-1} e_m - (q + q^{-1}) e_m e_{m-1} e_{m+1} e_m = 0, \\
&f_m f_{m-1} f_m f_{m+1} + f_{m-1} f_m f_{m+1} f_m + f_m f_{m+1} f_m f_{m-1} + \\
&\quad + f_{m+1} f_m f_{m-1} f_m - (q + q^{-1}) f_m f_{m-1} f_{m+1} f_m = 0,
\end{aligned} \tag{2.11}$$

one has, for $U_q(sl(m|n)^{(1)})$, the following extra q -Serre relations involving e_0 and f_0

$$\begin{aligned}
&e_0 e_1 e_0 e_{m+n-1} + e_1 e_0 e_{m+n-1} e_0 + e_0 e_{m+n-1} e_0 e_1 + \\
&\quad + e_{m+n-1} e_0 e_1 e_0 - (q + q^{-1}) e_0 e_1 e_{m+n-1} e_0 = 0, \\
&f_0 f_1 f_0 f_{m+n-1} + f_1 f_0 f_{m+n-1} f_0 + f_0 f_{m+n-1} f_0 f_1 + \\
&\quad + f_{m+n-1} f_0 f_1 f_0 - (q + q^{-1}) f_0 f_1 f_{m+n-1} f_0 = 0.
\end{aligned} \tag{2.12}$$

Definition 1 We set $U_q(sl(m|n)^{(1)})$ to be the Hopf superalgebra generated by the defining relations (2.3) of $\overline{U_q(sl(m|n)^{(1)})}$ subject to the extra q -Serre relations (2.11) and (2.12).

2.2 Extra q -Serre relations for $U_q(osp(2|2n)^{(1)})$

The set of simple roots of $osp(2|2n)^{(1)}$ can be expressed as

$$\begin{aligned}
\alpha_0 &= \delta - \epsilon - \delta_1, \\
\alpha_1 &= \epsilon - \delta_1, \\
\alpha_i &= \delta_{i-1} - \delta_i, & i &= 2, 3, \dots, n \\
\alpha_{n+1} &= 2\delta_n
\end{aligned} \tag{2.13}$$

with δ , ϵ and $\{\delta_i\}_{i=1}^n$ satisfying

$$\begin{aligned} (\delta, \delta) &= (\delta, \epsilon) = (\delta, \delta_i) = 0, & (\epsilon, \epsilon) &= -1, \\ (\delta_i, \delta_j) &= \delta_{ij}, & (\epsilon, \delta_i) &= 0. \end{aligned} \quad (2.14)$$

The Dynkin diagram for $osp(2|2n)^{(1)}$ is

$$(2.15)$$

From the Dynkin diagram we immediately see that $\overline{U_q(osp(2|2n))}$ is a subalgebra of $\overline{U_q(osp(2|2n)^{(1)})}$. Furthermore, there is no extra q -Serre relations for $U_q(osp(2|2n))$; namely, $U_q(osp(2|2n)) \equiv \overline{U_q(osp(2|2n))}$. However, for $U_q(osp(2|2n)^{(1)})$, we have found the following extra q -Serre relations:

$$\begin{aligned} X_q &\equiv e_2 e_1 e_0 + e_1 e_0 e_2 - e_2 e_0 e_1 - e_0 e_1 e_2 + (q + q^{-1})(e_0 e_2 e_1 - e_1 e_2 e_0) = 0, \\ Y_q &\equiv f_2 f_1 f_0 + f_1 f_0 f_2 - f_2 f_0 f_1 - f_0 f_1 f_2 + (q + q^{-1})(f_0 f_2 f_1 - f_1 f_2 f_0) = 0. \end{aligned} \quad (2.16)$$

Direct calculation shows that X_q , Y_q generate a proper Hopf algebra ideal \tilde{M}_q .

Definition 2 We set $U_q(osp(2|2n)^{(1)})$ to be the Hopf superalgebra generated by the defining relations (2.3) of $\overline{U_q(osp(2|2n)^{(1)})}$ subject to the extra q -Serre relations (2.16).

Note that the extra q -Serre relations (2.11), (2.12) and (2.16) have evident classical counterparts: in the $q \rightarrow 1$ limit they reduce to the extra Serre relations which need to be imposed to properly define the corresponding classical affine superalgebras. These extra Serre relations have also been found by Yamane [37]

2.3 The universal R-matrix

The algebras $U_q(sl(m|n)^{(1)})$ and $U_q(osp(2|2n)^{(1)})$ are quasitriangular graded Hopf algebras, which means the following:

Let Δ' be the opposite coproduct: $\Delta' = T\Delta$, where T is the graded twist map: $T(a \otimes b) = (-1)^{[a][b]} b \otimes a$, $\forall a, b \in U_q(\mathcal{G}^{(1)})$. Then Δ and Δ' are related by the universal R-matrix R in $U_q(\mathcal{G}^{(1)}) \otimes U_q(\mathcal{G}^{(1)})$ satisfying, among others, the relations

$$R\Delta(a) = \Delta'(a)R, \quad \forall a \in U_q(\mathcal{G}^{(1)}) \quad (2.17)$$

$$(I \otimes \Delta)R = R_{13}R_{12}, \quad (\Delta \otimes I)R = R_{13}R_{23} \quad (2.18)$$

where if $R = \sum a_i \otimes b_i$ then $R_{12} = \sum a_i \otimes b_i \otimes 1$, $R_{13} = \sum a_i \otimes 1 \otimes b_i$ etc. It follows from (2.18) that R satisfies the QYBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (2.19)$$

The multiplication rule for the tensor product is defined for elements $a, b, c, d \in U_q(\mathcal{G}^{(1)})$ by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd) \quad (2.20)$$

The existence of the universal R-matrix follows from Drinfeld's double construction [13, 14] provided the algebra has a triangular decomposition. This triangular decomposition follows from Theorem 1 in [35].

For any $x \in \mathbf{C}^\times$, we define an automorphism D_x of $U_q(\mathcal{G}^{(1)})$ as

$$D_x(e_i) = x^{\delta_{i0}} e_i, \quad D_x(f_i) = x^{-\delta_{i0}} f_i, \quad D_x(h_i) = h_i. \quad (2.21)$$

The parameter x is called the spectral parameter. One obtains a spectral parameter dependent universal R-matrix $R(x)$ in $U_q(\mathcal{G}^{(1)}) \otimes U_q(\mathcal{G}^{(1)})$ by setting

$$R(x) = (D_x \otimes I)(R). \quad (2.22)$$

It solves the spectral parameter dependent Yang-Baxter equation

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x). \quad (2.23)$$

3 General Solution of Jimbo's Equations for the R-matrices

In this section we will present our general technique for determining the spectral parameter dependent R-matrices for quantum superalgebras. From now on, with an abuse of notation, we will set

$$e_0 \equiv f_\psi, \quad f_0 \equiv e_\psi, \quad h_0 \equiv -h_\psi \quad (3.1)$$

Let π_λ , π_μ and π_ν be three irreps of a quantum superalgebra $U_q(\mathcal{G})$, afforded by the irreducible modules $V(\lambda)$, $V(\mu)$ and $V(\nu)$ with highest weights λ , μ and ν , respectively. Assume that all π_λ , π_μ and π_ν are affinizable, i.e. they can be extended to finite dimensional irreps of the corresponding quantum affine superalgebra $U_q(\mathcal{G}^{(1)})$. Let

$$R^{\lambda\mu}(x) = (\pi_\lambda \otimes \pi_\mu)(R(x)) \quad (3.2)$$

where $x \in \mathbf{C}$ is the spectral parameter introduced in eq. (2.22). Then $R^{\lambda\mu}(x)$ satisfies the system of linear equations [20] deduced from the intertwining property (2.17)

$$\begin{aligned} R^{\lambda\mu}(x) \Delta^{\lambda\mu}(a) &= \Delta'^{\lambda\mu}(a) R^{\lambda\mu}(x), \quad \forall a \in U_q(\mathcal{G}), \\ R^{\lambda\mu}(x) \left(x \pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2}) + \pi_\lambda(q^{h_0/2}) \otimes \pi_\mu(e_0) \right) \\ &= \left(x \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right) R^{\lambda\mu}(x) \end{aligned} \quad (3.3)$$

and satisfies the QYBE in the tensor product module $V(\lambda) \otimes V(\mu) \otimes V(\nu)$ of three irreps:

$$R_{12}^{\lambda\mu}(x)R_{13}^{\lambda\nu}(xy)R_{23}^{\mu\nu}(y) = R_{23}^{\mu\nu}(y)R_{13}^{\lambda\nu}(xy)R_{12}^{\lambda\mu}(x). \quad (3.4)$$

In the above, $\Delta^{\lambda\mu}(a) \equiv (\pi_\lambda \otimes \pi_\mu)\Delta(a)$. The solution $R^{\lambda\mu}(x)$ to the above equations intertwines the coproduct and opposite coproduct of $U_q(\mathcal{G}^{(1)})$ in the representation $(\pi_\lambda \otimes \pi_\mu)(\text{ev}_x \otimes \text{ev}_1)\Delta$. Because this representation is irreducible for generic x , by Schur's lemma the solution is uniquely determined up to a scalar factor. We fix this factor in such a way that

$$\check{R}^{\lambda\mu}(x)\check{R}^{\mu\lambda}(x^{-1}) = I, \quad (3.5)$$

which is usually called the unitarity condition in the literature. The multiplicative spectral parameter x can be transformed into an additive spectral parameter u by $x = \exp(u)$.

In all our equations we implicitly use the “graded” multiplication rule of eq. (2.20). Thus the R -matrix of a quantum superalgebra satisfies a “graded” Yang-Baxter equation which, when written as an ordinary matrix equation, contains extra signs:

$$\begin{aligned} & \left(R^{\lambda\mu}(x)\right)_{\alpha\beta}^{\alpha'\beta'} \left(R^{\lambda\nu}(xy)\right)_{\alpha'\gamma}^{\alpha''\gamma'} \left(R^{\mu\nu}(y)\right)_{\beta'\gamma'}^{\beta''\gamma''} (-1)^{[\alpha][\beta]+[\gamma][\alpha']+[\gamma'][\beta']} \\ &= \left(R^{\mu\nu}(y)\right)_{\beta'\gamma}^{\beta'\gamma'} \left(R^{\lambda\nu}(xy)\right)_{\alpha'\gamma'}^{\alpha'\gamma''} \left(R^{\lambda\mu}(x)\right)_{\alpha'\beta'}^{\alpha''\beta''} (-1)^{[\beta][\gamma]+[\gamma'][\alpha]+[\beta'][\alpha']}, \end{aligned} \quad (3.6)$$

where $[\alpha]$ denotes the degree of the basis vector v_α . However after a redefinition

$$\left(\tilde{R}^{\lambda\mu}\right)_{\alpha\beta}^{\alpha'\beta'} = \left(R^{\lambda\mu}\right)_{\alpha\beta}^{\alpha'\beta'} (-1)^{[\alpha][\beta]} \quad (3.7)$$

the signs disappear from the equation. Thus any solution of the “graded” Yang-Baxter equation arising from the R -matrix of a quantum superalgebra provides also a solution of the standard Yang-Baxter equation after the redefinition in eq. (3.7). The graded Yang-Baxter equation was studied already in [27].

Now introduce the graded permutation operator $P^{\lambda\mu}$ on the tensor product module $V(\lambda) \otimes V(\mu)$ such that

$$P^{\lambda\mu}(v_\alpha \otimes v_\beta) = (-1)^{[\alpha][\beta]} v_\beta \otimes v_\alpha, \quad \forall v_\alpha \in V(\lambda), \quad v_\beta \in V(\mu) \quad (3.8)$$

and set

$$\check{R}^{\lambda\mu}(x) = P^{\lambda\mu} R^{\lambda\mu}(x). \quad (3.9)$$

Then (3.3) can be rewritten as

$$\begin{aligned} & \check{R}^{\lambda\mu}(x)\Delta^{\lambda\mu}(a) = \Delta^{\mu\lambda}(a)\check{R}^{\lambda\mu}(x), \quad \forall a \in U_q(\mathcal{G}), \\ & \check{R}^{\lambda\mu}(x) \left(x\pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2}) + \pi_\lambda(q^{h_0/2}) \otimes \pi_\mu(e_0) \right) \\ &= \left(\pi_\mu(e_0) \otimes \pi_\lambda(q^{-h_0/2}) + x\pi_\mu(q^{h_0/2}) \otimes \pi_\lambda(e_0) \right) \check{R}^{\lambda\mu}(x) \end{aligned} \quad (3.10)$$

and in terms of $\check{R}^{\lambda\mu}(x)$ the QYBE becomes

$$(I \otimes \check{R}^{\lambda\mu}(x))(\check{R}^{\lambda\nu}(xy) \otimes I)(I \otimes \check{R}^{\mu\nu}(y)) = (\check{R}^{\mu\nu}(y) \otimes I)(I \otimes \check{R}^{\lambda\nu}(xy))(\check{R}^{\lambda\mu}(x) \otimes I) \quad (3.11)$$

both sides of which act from $V(\lambda) \otimes V(\mu) \otimes V(\nu)$ to $V(\nu) \otimes V(\mu) \otimes V(\lambda)$. Note that this equation, if written in matrix form, does not have extra signs in the superalgebra case. This is because the definition of the graded permutation operator in eq. (3.8) includes the signs of eq. (3.7).

Consider three special cases: $x = 0$, $x = \infty$ and $x = 1$. For these special values of x , $\check{R}^{\lambda\mu}(x)$ satisfies the spectral-free QYBE,

$$(I \otimes \check{R}^{\lambda\mu})(\check{R}^{\lambda\nu} \otimes I)(I \otimes \check{R}^{\mu\nu}) = (\check{R}^{\mu\nu} \otimes I)(I \otimes \check{R}^{\lambda\nu})(\check{R}^{\lambda\mu} \otimes I). \quad (3.12)$$

Moreover, from (3.10), we have respectively, for $x = 0$,

$$\begin{aligned} \check{R}^{\lambda\mu}(0)\Delta^{\lambda\mu}(a) &= \Delta^{\mu\lambda}(a)\check{R}^{\lambda\mu}(0), \quad \forall a \in U_q(\mathcal{G}), \\ \check{R}^{\lambda\mu}(0) \left(\pi_\lambda(q^{h_0/2}) \otimes \pi_\mu(e_0) \right) &= \left(\pi_\mu(e_0) \otimes \pi_\lambda(q^{-h_0/2}) \right) \check{R}^{\lambda\mu}(0) \end{aligned} \quad (3.13)$$

for $x = \infty$,

$$\begin{aligned} \check{R}^{\lambda\mu}(\infty)\Delta^{\lambda\mu}(a) &= \Delta^{\mu\lambda}(a)\check{R}^{\lambda\mu}(\infty), \quad \forall a \in U_q(\mathcal{G}), \\ \check{R}^{\lambda\mu}(\infty) \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2}) \right) &= \left(\pi_\mu(q^{h_0/2}) \otimes \pi_\lambda(e_0) \right) \check{R}^{\lambda\mu}(\infty) \end{aligned} \quad (3.14)$$

and for $x = 1$,

$$\begin{aligned} \check{R}^{\lambda\mu}(1)\Delta^{\lambda\mu}(a) &= \Delta^{\mu\lambda}(a)\check{R}^{\lambda\mu}(1), \quad \forall a \in U_q(\mathcal{G}), \\ \check{R}^{\lambda\mu}(1) \left(\pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2}) + \pi_\lambda(q^{h_0/2}) \otimes \pi_\mu(e_0) \right) &= \left(\pi_\mu(e_0) \otimes \pi_\lambda(q^{-h_0/2}) + \pi_\mu(q^{h_0/2}) \otimes \pi_\lambda(e_0) \right) \check{R}^{\lambda\mu}(1). \end{aligned} \quad (3.15)$$

Eqs.(3.13), (3.14) and (3.15) respectively admit a unique solution for any given two irreps of $U_q(\mathcal{G})$. We consider only the case where the tensor product decomposition

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} V(\nu), \quad (3.16)$$

is multiplicity-free.

We first focus on the $x = 1$ case, i.e. (3.15). In this case we have, from (3.5),

$$\check{R}^{\lambda\mu}(1)\check{R}^{\mu\lambda}(1) = I. \quad (3.17)$$

Let $\mathcal{P}_\nu^{\lambda\mu} : V(\lambda) \otimes V(\mu) \rightarrow V(\nu)$ be the projection operators which satisfy

$$\mathcal{P}_\nu^{\lambda\mu} \mathcal{P}_{\nu'}^{\lambda\mu} = \delta_{\nu\nu'} \mathcal{P}_\nu^{\lambda\mu}, \quad \sum_{\nu} \mathcal{P}_\nu^{\lambda\mu} = I. \quad (3.18)$$

Following [10], we define operators $\mathbf{P}_\nu^{\lambda\mu}$ by

$$\mathbf{P}_\nu^{\lambda\mu} = \mathcal{P}_\nu^{\mu\lambda} \check{R}^{\lambda\mu}(1) = \check{R}^{\lambda\mu}(1) \mathcal{P}_\nu^{\lambda\mu}. \quad (3.19)$$

Then, by definition

$$\check{R}^{\lambda\mu}(1) = \sum_{\nu} \mathbf{P}_\nu^{\lambda\mu}. \quad (3.20)$$

It is easy to show that

$$\mathbf{P}_\nu^{\lambda\mu} \mathcal{P}_{\nu'}^{\lambda\mu} = \mathcal{P}_{\nu'}^{\mu\lambda} \mathbf{P}_\nu^{\lambda\mu} = \delta_{\nu\nu'} \mathbf{P}_\nu^{\lambda\mu}, \quad (3.21)$$

and

$$\begin{aligned} \mathbf{P}_\nu^{\mu\lambda} \mathbf{P}_{\nu'}^{\lambda\mu} &= \mathcal{P}_\nu^{\lambda\mu} \check{R}^{\mu\lambda}(1) \check{R}^{\lambda\mu}(1) \mathcal{P}_{\nu'}^{\lambda\mu} \\ &= \mathcal{P}_\nu^{\lambda\mu} \mathcal{P}_{\nu'}^{\lambda\mu} = \delta_{\nu\nu'} \mathcal{P}_\nu^{\lambda\mu} \end{aligned} \quad (3.22)$$

Eqs.(3.21, 3.22) imply that the operators $\mathbf{P}_\nu^{\lambda\mu}$ are “projection” operators. As can be seen from (3.15), the “projectors” $\mathbf{P}_\nu^{\lambda\mu}$ are intertwiners of $U_q(\mathcal{G})$. The general solution satisfying the first equation of (3.10) thus is a sum of these elementary intertwiners

$$\check{R}^{\lambda\mu}(x) = \sum_\nu \rho_\nu(x) \mathbf{P}_\nu^{\lambda\mu} \quad (3.23)$$

where $\rho_\nu(x)$ are some functions of x . Our task is now to determine these functions so that (3.23) satisfies also the second equation of (3.10).

We begin by determining $\rho_\nu(x)$ at the special value $x = 0$. At $x = 0$ the R-matrix $\check{R}^{\lambda\mu}(0)$ reduces to the R-matrix of the quantum superalgebra $U_q(\mathcal{G})$ in $V(\lambda) \otimes V(\mu)$: $\check{R}^{\lambda\mu}(0) \equiv \check{R}^{\lambda\mu}$. We recall the following fact about the universal R-matrix \bar{R} for $U_q(\mathcal{G})$:

$$\bar{R}^T \bar{R} = (v \otimes v) \Delta(v^{-1}), \quad v = uq^{-2h_\rho}, \quad \pi_\lambda(v) = q^{-C(\lambda)} \cdot I_{V(\lambda)} \quad (3.24)$$

with $u = \sum_i (-1)^{[i]} S(\bar{b}_i) \bar{a}_i$ where $\bar{R} = \sum_i \bar{a}_i \otimes \bar{b}_i$. S is the antipode for $U_q(\mathcal{G})$, $C(\lambda) = (\lambda, \lambda + 2\rho)$ is the eigenvalue of the quadratic Casimir element of \mathcal{G} in an irrep with highest weight λ ; ρ is the graded half-sum of positive roots of \mathcal{G} . We now compute $\check{R}^{\mu\lambda} \check{R}^{\lambda\mu}$. Using the above equations we have

$$\begin{aligned} \sum_\nu (\rho_\nu(0))^2 \mathcal{P}_\nu^{\lambda\mu} &= \check{R}^{\mu\lambda} \check{R}^{\lambda\mu} \equiv P^{\mu\lambda} \bar{R}^{\mu\lambda} P^{\lambda\mu} \bar{R}^{\lambda\mu} \\ &= (\bar{R}^T)^{\lambda\mu} \bar{R}^{\lambda\mu} = (\pi_\lambda \otimes \pi_\mu)(\bar{R}^T \bar{R}) \\ &= \pi_\lambda(v) \otimes \pi_\mu(v) (\pi_\lambda \otimes \pi_\mu) \Delta(v^{-1}) \\ &= \sum_\nu q^{C(\nu) - C(\lambda) - C(\mu)} \mathcal{P}_\nu^{\lambda\mu} \end{aligned} \quad (3.25)$$

where use has been made of $\check{R}^{\lambda\mu} \equiv P^{\lambda\mu} \bar{R}^{\lambda\mu} \equiv P^{\lambda\mu} (\pi_\lambda \otimes \pi_\mu)(\bar{R})$. It follows immediately that

$$\rho_\nu(0) = \epsilon(\nu) q^{\frac{C(\nu) - C(\lambda) - C(\mu)}{2}} \quad (3.26)$$

where $\epsilon(\nu)$ is the *parity* of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$: it arises as an eigenvalue of the unitary self-adjoint operator $\check{R}^{\lambda\mu}(1)$ at $q = 1$ (see [10] for an explanation). Thus

$$\check{R}^{\lambda\mu}(0) = \sum_\nu \epsilon(\nu) q^{\frac{C(\nu) - C(\lambda) - C(\mu)}{2}} \mathbf{P}_\nu^{\lambda\mu}. \quad (3.27)$$

A similar relation for quantum bosonic algebras was obtained in [32, 15].

The $\rho_\nu(x)$ at the special value $x = \infty$ can be easily obtained from (3.27) with the help of the unitarity condition (3.5):

$$\check{R}^{\lambda\mu}(\infty) = \sum_{\nu} \epsilon(\nu) q^{-\frac{C(\nu)-C(\lambda)-C(\mu)}{2}} \mathbf{P}_{\nu}^{\lambda\mu}. \quad (3.28)$$

Remark: both $\mathcal{P}_{\nu}^{\lambda\mu}$ and $\mathbf{P}_{\nu}^{\lambda\mu}$ can be determined by using pure representation theory of $U_q(\mathcal{G})$ as follows. Let $\{|e_{\alpha}^{\nu}\rangle_{\lambda\otimes\mu}\}$ be an orthonormal basis for $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. $V(\nu)$ is also embedded in $V(\mu) \otimes V(\lambda)$ through the opposite coproduct Δ' . Let $\{|e_{\alpha}^{\nu}\rangle_{\mu\otimes\lambda}\}$ be the corresponding orthonormal basis. Then for real generic $q > 0$, $\mathcal{P}_{\nu}^{\lambda\mu}$ and $\mathbf{P}_{\nu}^{\lambda\mu}$ may be written as

$$\begin{aligned} \mathcal{P}_{\nu}^{\lambda\mu} &= \sum_{\alpha} |e_{\alpha}^{\nu}\rangle_{\lambda\otimes\mu} {}_{\lambda\otimes\mu}\langle e_{\alpha}^{\nu}|, \\ \mathbf{P}_{\nu}^{\lambda\mu} &= \sum_{\alpha} |e_{\alpha}^{\nu}\rangle_{\mu\otimes\lambda} {}_{\lambda\otimes\mu}\langle e_{\alpha}^{\nu}|. \end{aligned} \quad (3.29)$$

which should extend to all complex q via analytic continuation arguments. For more details see [11].

Now we have sufficient relations to solve (3.10) generally. Inserting (3.23) into the second equation of (3.10), and using (3.13), (3.14), (3.15) and the spectral decomposition formulae (3.20), (3.27) and (3.28), one ends up with [10]

$$\begin{aligned} &\left\{ \rho_{\nu}(x) \left(xq^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu')q^{C(\nu')/2} \right) - \rho_{\nu'}(x) \left(q^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu')xq^{C(\nu')/2} \right) \right\} \\ &\quad \times \mathcal{P}_{\nu}^{\lambda\mu} \left(\pi_{\lambda}(e_0) \otimes \pi_{\mu}(q^{-h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu} = 0, \quad \forall \nu \neq \nu'. \end{aligned} \quad (3.30)$$

If

$$\mathcal{P}_{\nu}^{\lambda\mu} \left(\pi_{\lambda}(e_0) \otimes \pi_{\mu}(q^{-h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu} \neq 0 \quad (3.31)$$

then (3.30) gives rise to a relation between $\rho_{\nu}(x)$ and $\rho_{\nu'}(x)$,

$$\rho_{\nu}(x) = \rho_{\nu'}(x) \frac{q^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu')xq^{C(\nu')/2}}{xq^{C(\nu)/2} + \epsilon(\nu)\epsilon(\nu')q^{C(\nu')/2}}, \quad \forall \nu \neq \nu'. \quad (3.32)$$

It can be shown [10] that $\epsilon(\nu)\epsilon(\nu') = -1$ always if (3.31) is satisfied. With the help of notation

$$\langle a \rangle \equiv \frac{1 - xq^a}{x - q^a} \quad (3.33)$$

(3.32) then becomes

$$\rho_{\nu}(x) = \left\langle \frac{C(\nu') - C(\nu)}{2} \right\rangle \rho_{\nu'}(x). \quad (3.34)$$

We have a relation between the coefficients ρ_{ν} and $\rho_{\nu'}$ whenever the condition eq. (3.31) is satisfied, i.e., whenever $\pi_{\lambda}(e_0) \otimes \pi_{\mu}(q^{-h_0/2})$ maps from the module $V(\nu')$ to the module $V(\nu)$. As a graphical aid [38] we introduce the tensor product graph.

Definition 3 *The tensor product graph $G^{\lambda\mu}$ associated to the tensor product $V(\lambda) \otimes V(\mu)$ is a graph whose vertices are the irreducible modules $V(\nu)$ appearing in the decomposition eq. (3.16) of $V(\lambda) \otimes V(\mu)$. There is an edge between a vertex $V(\nu)$ and a vertex $V(\nu')$ iff*

$$\mathcal{P}_{\nu}^{\lambda\mu} \left(\pi_{\lambda}(e_0) \otimes \pi_{\mu}(q^{-h_0/2}) \right) \mathcal{P}_{\nu'}^{\lambda\mu} \neq 0. \quad (3.35)$$

If $V(\lambda)$ and $V(\mu)$ are irreducible $U_q(\mathcal{G})$ -modules then the tensor product graph is always connected, i.e., every node is linked to every other node by a path of edges. This follows from the fact that $\pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2})$ is related to the lowest component of an adjoint tensor operator; for details see [38]. This implies that the relations eq. (3.34) are sufficient to determine all the coefficients $\rho_\nu(x)$ uniquely. If the tensor product graph is multiply connected, i.e., if there exist more than two paths between two nodes, then the relations overdetermine the coefficients, i.e., there are consistency conditions. However, because the existence of a solution to the Jimbo equations is guaranteed by the existence of the universal R-matrix, these consistency conditions will always be satisfied.

The straightforward but tedious and impractical way to determine the tensor product graph is to work out explicitly the left hand side of (3.35). It is much more practical to work instead with the following larger graph.

Definition 4 *The extended tensor product graph $\Gamma^{\lambda\mu}$ associated to the tensor product $V(\lambda) \otimes V(\mu)$ is a graph whose vertices are the irreducible modules $V(\nu)$ appearing in the decomposition eq. (3.16) of $V(\lambda) \otimes V(\mu)$. There is an edge between two vertices $V(\nu)$ and $V(\nu')$ iff*

$$V(\nu') \subset V_{adj} \otimes V(\nu) \quad \text{and} \quad \epsilon(\nu)\epsilon(\nu') = -1. \quad (3.36)$$

It follows again from the fact that $\pi_\lambda(e_0) \otimes \pi_\mu(q^{-h_0/2})$ is related to the lowest component of an adjoint tensor operator that the condition in eq. (3.36) is a necessary condition for eq. (3.35) [38]. This means that every link contained in the tensor product graph is contained also in the extended tensor product graph but the latter may contain more links. Only if the extended tensor product graph is a tree do we know that it is equal to the tensor product graph. If we impose a relation (3.34) on the ρ 's for every link in the extended tensor product graph, we may be imposing too many relations and thus may not always find a solution. If however we do find a solution, then this is the unique correct solution which we would have obtained also from the tensor product graph.

The advantage of using the extended tensor product graph is that it can be constructed using only Lie algebra representation theory. We only need to be able to decompose tensor products and to determine the parity of submodules.

4 New R-Matrices

We will now apply the technique developed in the last section to a number of interesting examples.

4.1 Examples of R-Matrices for $U_q(sl(m|n)^{(1)})$

As already seen in section 2.1 we find it convenient to embed the root space of $sl(m|n)$ into the bigger space spanned by $\{\epsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n$ and equipped with the non-definite inner product

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\delta_i, \delta_j) = -\delta_{ij}, \quad (\epsilon_i, \delta_j) = 0, \quad (4.1)$$

The root space is the subspace with no component in the direction of $\sum_{i=1}^n \delta_i$. Any weight Λ may be written as

$$\Lambda \equiv (\Lambda_1, \dots, \Lambda_m | \bar{\Lambda}_1, \dots, \bar{\Lambda}_n) \equiv \sum_{i=1}^m \Lambda_i \epsilon_i + \sum_{j=1}^n \bar{\Lambda}_j \delta_j. \quad (4.2)$$

The graded half sum ρ of the positive roots of $sl(m|n)$ is

$$2\rho = \sum_{i=1}^m (m - n - 2i + 1) \epsilon_i + \sum_{j=1}^n (m + n - 2j + 1) \delta_j. \quad (4.3)$$

For convenience of notation we define the following elements

$$\lambda_b = \sum_{i=1}^b \epsilon_i. \quad (4.4)$$

Because an evaluation homomorphism exists for $U_q(sl(m|n)^{(1)})$ [39][¶], every irrep of $U_q(sl(m|n))$ provides also an irrep for $U_q(sl(m|n)^{(1)})$. Thus to any tensor product of two irreps of $U_q(sl(m|n))$ there corresponds a spectral parameter dependent R-matrix. Here, as examples, we will construct the R-matrices corresponding to the following tensor products: (i) rank a (≥ 1) symmetric tensor (which has the highest weight $a\lambda_1$) with rank b (≥ 1) symmetric tensor of the same type; (ii) rank a antisymmetric tensor with rank b antisymmetric tensor of the same type. Without loss of generality, we assume $m \geq a \geq b$ in the following. For more information on the irreducible representations see Appendix B

We start with case (i). We assume the symmetric tensors to be contravariant. The case of covariant symmetric tensors can be similarly treated. Such contravariant symmetric tensors are obviously atypical and type (1) unitary. In view of arguments in Appendix B we obtain the tensor product decomposition

$$V(a\lambda_1) \otimes V(b\lambda_1) = \bigoplus_c V(\Lambda_c), \quad (4.5)$$

where

$$\Lambda_c = (a + b - c)\epsilon_1 + c\epsilon_2, \quad c = 0, 1, \dots, b. \quad (4.6)$$

The corresponding tensor product graph is

$$V(a\lambda_1) \otimes V(b\lambda_1) = \begin{array}{c} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \Lambda_0 \quad \Lambda_1 \quad \quad \quad \Lambda_{b-1} \quad \Lambda_b \end{array} \quad (4.7)$$

[¶]In [39] the author constructs an evaluation homomorphism from $\overline{U_q(sl(m|n)^{(1)})}$ to $U_q(sl(m|n))$. However, one can show that the evaluation map constructed in [39] also preserves the extra q -Serre relations (2.12) and thus also provides a homomorphism from $U_q(sl(m|n)^{(1)})$ to $U_q(sl(m|n))$.

The Casimir takes the following values on the representations appearing in the above graph

$$C(\Lambda_c) = (a + b - c)^2 + c^2 - 2c + (a + b)(m - n - 1). \quad (4.8)$$

Using this information and eq. (3.34) we can determine all the functions $\rho_\nu(x)$ and we arrive at the spectral decomposition of the R-matrix

$$\check{R}^{a\lambda_1, b\lambda_1}(x) = \rho_{\Lambda_0}(x) \sum_{c=0}^b \prod_{i=1}^c \langle a + b - 2i + 2 \rangle \mathbf{P}_{\Lambda_c}^{a\lambda_1, b\lambda_1} \quad (4.9)$$

(our convention here and below is that $\prod_{i=1}^0(\dots) = 1$). The $a = b = 1$ case had been worked out already in [3, 4]. The overall scalar factor $\rho_{\Lambda_0}(x)$ is not determined by the Jimbo equations or by the Yang-Baxter equation but by the normalization condition eq. (3.5). We will from now on drop such overall scalar factors from the formulas for the R-matrices.

It should be emphasized that although the spectral decomposition (4.9) of the R-matrices obtained from $U_q(sl(m|n))$ formally look like those for the R-matrices obtained from $U_q(sl(m+n))$, they are actually completely different. For example the ranks of the elementary intertwiners $\mathbf{P}_{\Lambda_c}^{a\lambda_1, b\lambda_1}$ in (4.9) differ from those in the case of $U_q(sl(m+n))$ and thus the R-matrices (4.9) have different multiplicities of eigenvalues from those for $U_q(sl(m+n))$. In particular, the R-matrices for the case of $a = b = 1$ are known to give rise to the Perk-Schultz model R-matrices [8, 30]. As an example, let us compute the projectors for the $a = b = 1$ case of $U_q(sl(2|1))$ explicitly. We have,

$$\begin{aligned} \mathbf{P}_{\Lambda_0}^{\lambda_1 \lambda_1} &= [2]_q^{-1} \left([2]_q (e_{11} + e_{55}) + q(e_{22} + e_{33} + e_{66}) + q^{-1}(e_{44} + e_{77} + e_{88}) \right. \\ &\quad \left. + e_{24} + e_{37} + e_{42} + e_{68} + e_{73} + e_{86} \right) \\ \mathbf{P}_{\Lambda_1}^{\lambda_1 \lambda_1} &= 1 - \mathbf{P}_{\Lambda_0}^{\lambda_1 \lambda_1} \end{aligned} \quad (4.10)$$

and the R-matrix is

$$\begin{aligned} \check{R}^{\lambda_1 \lambda_1}(x) &= e_{11} + e_{55} + \frac{1 - xq^2}{x - q^2} e_{99} + \frac{1 - q^{-2}}{1 - xq^{-2}} (e_{22} + e_{33} + e_{66} + x(e_{44} + e_{77} + e_{88})) \\ &\quad + \frac{q^{-1}(1 - x)}{1 - xq^{-2}} (e_{24} + e_{37} + e_{68} + e_{42} + e_{73} + e_{86}) \end{aligned} \quad (4.11)$$

where e_{ij} is the matrix satisfying $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$. (4.11) is the Perk-Schultz 15-vertex model R-matrix.

For case (ii) we again assume the antisymmetric tensors to be contravariant. Such contravariant antisymmetric tensors are atypical and type (1) unitary. Following the arguments of Appendix B, the tensor product decomposition can be worked out to be

$$V(\lambda_a) \otimes V(\lambda_b) = \bigoplus_c V(\Lambda_c) \quad (4.12)$$

where, when $a + b \leq m$,

$$\Lambda_c = \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i, \quad c = 0, 1, \dots, b \quad (4.13)$$

and when $a + b > m$,

$$\begin{aligned}\Lambda_c &= \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i, & c = 0, 1, \dots, m-a \\ \Lambda_c &= \sum_{i=1}^m \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i + (a+c-m)\delta_1, & c = m-a+1, \dots, b\end{aligned}\quad (4.14)$$

The corresponding tensor product graph is

$$V(\lambda_a) \otimes V(\lambda_b) = \begin{array}{c} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \Lambda_0 \quad \Lambda_1 \quad \quad \quad \Lambda_{b-1} \quad \Lambda_b \end{array} \quad (4.15)$$

The Casimirs are

$$C(\Lambda_c) = 2c(b-a-1) - 2c^2 + (m-n+1)(a+b) + 2b - a^2 - b^2 \quad (4.16)$$

and we obtain the following spectral decomposition of the R-matrix

$$\check{R}^{\lambda_a, \lambda_b}(x) = \sum_{c=0}^b \prod_{i=1}^c \langle 2i + a - b \rangle \mathbf{P}_{\Lambda_c}^{\lambda_a, \lambda_b} \quad (4.17)$$

4.2 R-Matrices for $U_q(sl(1|n)^{(1)})$ with Extra Continuous Parameters

It is well known [22] that type-I superalgebras admit nontrivial one-parameter families of finite-dimensional irreps which deform to provide one-parameter families of finite-dimensional irreps of the corresponding type-I quantum superalgebras [17].

For an indeterminate $\alpha \in \mathbf{R}$ we will consider the one-parameter family of 2^n -dimensional irreducible $U_q(sl(1|n))$ -modules $V(\alpha)$ with highest weights of the form $\Lambda(\alpha) = (\alpha|0, \dots, 0) \equiv \alpha\epsilon$. From the classification scheme given in Appendix B, $V(\alpha)$ is a unitary irreducible module provided that $0 < q \in \mathbf{R}$ and $\alpha > n-1$ or $\alpha < 0$. It follows that the tensor product module $V(\alpha) \otimes V(\beta)$ is completely reducible for $\alpha, \beta > n-1$ or $\alpha, \beta < 0$. The decomposition is given by

$$V(\alpha) \otimes V(\beta) = \bigoplus_{c=0}^n V(\Lambda_c) \quad (4.18)$$

with

$$\Lambda_c = (\alpha + \beta - c)\epsilon + \lambda_c, \quad \lambda_c = \sum_{i=1}^c \delta_i \quad (4.19)$$

To see this, we observe that each highest weight occurring in the decomposition is of the form $\alpha\epsilon + \mu$, where μ is an element of the weight spectrum of $V(\beta)$. Moreover, to ensure that $\alpha\epsilon + \mu$ is a dominant weight for $sl(1|n)$, the weight μ must be dominant for $u(1) \oplus sl(n)$. From the induced module construction of Appendix A, we see that μ must be of the form $\mu_c = (\beta - c)\epsilon + \lambda_c$ and it occurs with unit multiplicity. It is easily checked by dimensions that all highest weights $\Lambda_c = \alpha\epsilon + \mu_c$ arise in the decomposition, leading to (4.18).

The tensor product graph in this case is simply

$$V(\alpha) \otimes V(\beta) = \begin{array}{c} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \\ \Lambda_0 \quad \Lambda_1 \quad \quad \quad \Lambda_{n-1} \quad \Lambda_n \end{array} \quad (4.20)$$

Using the fact that $\Lambda_{c+1} = \Lambda_c - \epsilon + \delta_{c+1}$, we obtain the differences of the Casimirs

$$\frac{C(\Lambda_c) - C(\Lambda_{c+1})}{2} = \alpha + \beta - 2c. \quad (4.21)$$

For the R-matrix we find

$$\check{R}^{\alpha\beta}(x) = \sum_{c=0}^n \prod_{i=1}^c \langle \alpha + \beta - 2i + 2 \rangle \mathbf{P}_{\Lambda_c}^{\alpha\beta}. \quad (4.22)$$

This R-matrix is obtained for $0 < q \in \mathbb{R}$ and $\alpha, \beta > n - 1$ or $\alpha, \beta < 0$ but should hold for other values of q , α , β via analytic continuation arguments.

It is worth emphasizing that $\check{R}^{\alpha\beta}(x)$ satisfies the QYBE

$$(I \otimes \check{R}^{\alpha\beta}(x))(\check{R}^{\alpha\gamma}(xy) \otimes I)(I \otimes \check{R}^{\beta\gamma}(y)) = (\check{R}^{\beta\gamma}(y) \otimes I)(I \otimes \check{R}^{\alpha\gamma}(xy) \otimes I)(\check{R}^{\alpha\beta}(x) \otimes I) \quad (4.23)$$

acting on $V(\alpha) \otimes V(\beta) \otimes V(\gamma)$ and the parameters α , β and γ enter the QYBE in a non-additive form.

We should remark that although it appears that the eigenvalues of $\check{R}^{\alpha\beta}(x)$ depend only on $\alpha + \beta$, the elementary intertwiners $\mathbf{P}_{\Lambda_c}^{\alpha\beta}$ depend on both α and β . For example, for $U_q(sl(1|1))$ ^{||}, the $\mathbf{P}_{\Lambda_0}^{\alpha\beta}$ and $\mathbf{P}_{\Lambda_1}^{\alpha\beta}$ in the above equation take the form

$$\begin{aligned} \mathbf{P}_{\Lambda_0}^{\alpha\beta} &= [\alpha + \beta]_q^{-1} \begin{pmatrix} [\alpha + \beta]_q & 0 & 0 & 0 \\ 0 & ([\alpha]_q [\beta]_q)^{1/2} q^{(\alpha+\beta)/2} & [\alpha]_q & 0 \\ 0 & [\beta]_q & ([\alpha]_q [\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{P}_{\Lambda_1}^{\alpha\beta} &= [\alpha + \beta]_q^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & ([\alpha]_q [\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} & -[\beta]_q & 0 \\ 0 & -[\alpha]_q & ([\alpha]_q [\beta]_q)^{1/2} q^{(\alpha+\beta)/2} & 0 \\ 0 & 0 & 0 & [\alpha + \beta]_q \end{pmatrix} \end{aligned} \quad (4.24)$$

and the R-matrix in this case reads

$$\check{R}^{\alpha\beta}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -([\alpha]_q [\beta]_q)^{1/2} q^{(\alpha+\beta)/2} \cdot \frac{q-q^{-1}}{x-q^{\alpha+\beta}} & \frac{xq^\beta - q^\alpha}{x-q^{\alpha+\beta}} & 0 \\ 0 & \frac{xq^\alpha - q^\beta}{x-q^{\alpha+\beta}} & -([\alpha]_q [\beta]_q)^{1/2} q^{(\alpha+\beta)/2} \cdot \frac{x(q-q^{-1})}{x-q^{\alpha+\beta}} & 0 \\ 0 & 0 & 0 & \frac{1-xq^{\alpha+\beta}}{x-q^{\alpha+\beta}} \end{pmatrix} \quad (4.25)$$

More details can be found in [7]. It should be pointed out that (4.25) is the trigonometric limit, up to a similarity transformation (c.f. [5]), of the 8-vertex free fermion model R-matrix with extra non-additive parameters, obtained in [2].

^{||} $sl(1|1)$ is not a simple algebra, but our method give nevertheless a valid R-matrix, as the example shows.

5 Discussion

By extending the formalism of [10] for quantum bosonic algebras, a systematic method was developed for obtaining trigonometric solutions $\check{R}(x) \in \text{End}(V(\lambda) \otimes V(\mu))$ to the QYBE for the type-I quantum superalgebras $U_q(\mathcal{G})$, where $V(\lambda)$, $V(\mu)$ are irreducible $U_q(\mathcal{G})$ -modules such that their tensor product is completely reducible and multiplicity-free. In this connection, it is worth noting that the type-I quantum superalgebras admit two distinct (large) classes of unitary irreps, classified in refs.[17, 28] and discussed in Appendix B, so that $V(\lambda) \otimes V(\mu)$ is automatically completely reducible provided $V(\lambda)$, $V(\mu)$ are both unitary of the same type.

Our approach, which is based on the tensor product graph method, enables the construction of a large number of new R-matrices. As explicit examples, we have considered the cases where $V(\lambda)$, $V(\mu)$ correspond to the contravariant symmetric or anti-symmetric tensor irreps of $U_q(sl(m|n))$ in section 4.1. They give rise to generalizations of the Perk-Schultz model R-matrices.

As noted in the introduction, quantum superalgebras are not straightforward generalizations of quantum algebras but have a far richer structure and representation theory. In particular, the type-I quantum superalgebras have the intriguing property that they admit one-parameter families of irreps $V(\Lambda(\alpha))$, $\Lambda(\alpha) \equiv \Lambda + \alpha \sum_i \epsilon_i$, corresponding to each typical $\Lambda \in D_+$ (c.f. Appendix A). For real α sufficiently large these irreps are all unitary, typical and have the same dimension. This enables the construction of new R-matrices $\check{R}^{\alpha\beta}(x) \in \text{End}[V(\Lambda(\alpha)) \otimes V(\Lambda(\beta))]$ depending on two extra parameters α , β .

It should be emphasized that these R-matrices do not satisfy the usual QYBE (except when $\alpha = \beta$) but rather its natural extension, eq.(4.23), in which the parameters α , β enter in a way analogous to the spectral parameter x but in a non-additive form. Nevertheless, as will be shown elsewhere, solutions to this equation give rise to exactly solvable models and the entire apparatus of the QISM, including the (nested) algebraic Bethe ansatz, can be extended for the treatment of these models. In this way we obtain new three parameter exactly solvable models generalizing the (trigonometric) free fermion model [2]. The latter in fact arises from a certain one parameter family of two dimensional irreps of $U_q(sl(1|1))$, as we have shown in the paper.

Further new examples of such R-matrices arising from one-parameter families of irreps for $U_q(sl(1|n))$ (which includes the free fermion six vertex model when $n = 1$) were obtained in section 4.2. In future work we aim to investigate the exactly solvable models arising from these trigonometric R-matrices (and their rational limits) and the physical significance of the two extra parameters, α , β .

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A Kac's Induced Module Construction

For self-containedness, in this appendix we will give a brief overview of the induced module construction for type-I superalgebras due to Kac [21, 22]. However, as is shown in [40, 41], the construction also applies in the quantum case for generic q , giving modules with the same dimension and weight spectrum as the $q = 1$ case.

Let \mathcal{G} denote a Lie superalgebra with the distinguished \mathbf{Z} gradation [21],

$$\mathcal{G} = \bigoplus_{i \in \mathbf{Z}} \mathcal{G}_i \quad (\text{A.1})$$

satisfying $[\mathcal{G}_i, \mathcal{G}_j] = \mathcal{G}_{i+j}$. For type-I Lie superalgebras, (A.1) simplifies to $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$. This immediately implies that for $x, y \in \mathcal{G}_{-1}$, $[x, y] = xy + yx = 0 \implies xy = -yx$, and in particular, $x^2 = 0$.

Let Δ_1^+ denote the positive odd roots of \mathcal{G} . We have that $\{F_\alpha | \alpha \in \Delta_1^+\}$ form a basis for \mathcal{G}_{-1} and $\{E_\alpha | \alpha \in \Delta_1^+\}$ a basis for \mathcal{G}_1 . We refer to \mathcal{G}_0 as the “even subalgebra” of \mathcal{G} . Specifically, we have $\mathcal{G}_0 = u(1) \oplus sl(m) \oplus sl(n)$, for $\mathcal{G} = sl(m|n)$ with $m, n \geq 2$, $\mathcal{G}_0 = u(1) \oplus sl(n)$, for $\mathcal{G} = sl(1|n)$, $n \geq 2$ and $\mathcal{G}_0 = u(1) \oplus sp(2n)$, for $\mathcal{G} = osp(2|2n)$. From (A.1) we see that $[\mathcal{G}_0, \mathcal{G}_{-1}] = \mathcal{G}_{-1}$ so that \mathcal{G}_{-1} provides a representation of \mathcal{G}_0 . Let $U(\mathcal{G}_{-1})$ denote the universal enveloping algebra of \mathcal{G}_{-1} . $U(\mathcal{G}_{-1})$ is finite dimensional and the P.B.W theorem takes a particularly simple form. We choose the following basis elements for $U(\mathcal{G}_{-1})$

$$\left\{ \Gamma(\hat{t}) = \prod_{\alpha \in \Delta_1^+} (F_\alpha)^{t_\alpha}, \quad t_\alpha = 0, 1 \right\} \quad (\text{A.2})$$

which are unique up to a sign. These elements generate a finite dimensional module for \mathcal{G}_0 under the adjoint action.

Now let $V_0(\Lambda)$ denote an irreducible \mathcal{G}_0 -module and set $\overline{V(\Lambda)}$ to be the tensor product module

$$\overline{V(\Lambda)} = U(\mathcal{G}_{-1}) \otimes_{U(\mathcal{G}_0)} V_0(\Lambda). \quad (\text{A.3})$$

We refer to $\overline{V(\Lambda)}$ as the Kac module.

The \mathcal{G} -module $\overline{V(\Lambda)}$ is not necessarily irreducible. If it is, we set $V(\Lambda) = \overline{V(\Lambda)}$ and refer to Λ and $V(\Lambda)$ as “typical”. On the other hand, if $\overline{V(\Lambda)}$ contains a proper maximal submodule M (necessarily unique) we set $V(\Lambda) = \overline{V(\Lambda)} / M$ and refer to Λ and $V(\Lambda)$ as “atypical”. There is a well known criterion for typicality of $V(\Lambda)$ due to Kac [22], that is $V(\Lambda)$ is typical \mathcal{G} -module iff $(\Lambda + \rho, \alpha) \neq 0$, $\forall \alpha \in \Delta_1^+$.

Let us remark that for typical modules the dimensions are easily evaluated to be $\dim V(\Lambda) = 2^d \cdot \dim V_0(\Lambda)$, where d (which equals to mn for $sl(m|n)$ and to $2n$ for $osp(2|2n)$) is the number of odd positive roots. This formula is particularly useful in determining tensor product decompositions of typical modules.

The induced module construction provides an insight into the existence of one-parameter families of irreps. Observe that for the type-I superalgebras, the even subalgebra \mathcal{G}_0 contains a $u(1)$ subalgebra. Hence when we take the \mathcal{G}_0 -module $V_0(\Lambda)$, the action of the $u(1)$ subalgebra is simply scalar multiplication by an arbitrary $\alpha \in \mathbb{C}$. When we take the Kac module, we thereby obtain a one-parameter family of modules with the same dimension.

B Classification Theorems of Finite-Dimensional Irreps

The type-I superalgebras admit two types of unitary representations which may be described as follows. We define a conjugation operation on \mathcal{G} generators by $e_i^\dagger = f_i$, $f_i^\dagger = e_i$, $h_i^\dagger = h_i$ which is extended uniquely to all of \mathcal{G} such that $(uv)^\dagger = v^\dagger u^\dagger$, $\forall u, v \in \mathcal{G}$. If π_Λ denotes a representation of highest weight Λ then we call π_Λ type (1) unitary if

$$\pi_\Lambda(u^\dagger) = \overline{\pi_\Lambda(u)}, \quad \forall u \in \mathcal{G} \quad (\text{B.1})$$

and type (2) unitary if

$$\pi_\Lambda(u^\dagger) = (-1)^{[u]} \overline{\pi_\Lambda(u)}, \quad \forall u \in \mathcal{G} \quad (\text{B.2})$$

where the overline denotes Hermitian matrix conjugation. The two types of unitary representations are in fact related via duality.

Such unitary representations have the property that they are always completely reducible and the tensor product of two representations of the same type reduce completely into unitary representations of the same type.

The finite dimensional irreducible unitary representations have been classified in [16]. Reference [16] deals with the irreps of $gl(m|n)$ rather than $sl(m|n)$. There is however no substantial difference besides the fact that some irreps which are distinct as irreps of $gl(m|n)$ are isomorphic as irreps of $sl(m|n)$.

Theorem 1 *A given $gl(m|n)$ -module $V(\Lambda)$, with $\Lambda \in D_+$, is type (1) unitary iff: i) $(\Lambda + \rho, \epsilon_m - \delta_n) > 0$; or ii) there exists an odd index $\mu \in \{1, 2, \dots, n\}$ such that $(\Lambda + \rho, \epsilon_m - \delta_\mu) = 0 = (\Lambda, \delta_\mu - \delta_n)$. In the former case the given condition also enforces typicality on $V(\Lambda)$, while in the latter case all irreps are atypical.*

Theorem 2 *A given $gl(m|n)$ -module $V(\Lambda)$, with $\Lambda \in D_+$, is type (2) unitary iff: i) $(\Lambda + \rho, \epsilon_1 - \delta_1) < 0$; or ii) there exists an even index $k \in \{1, 2, \dots, m\}$ such that $(\Lambda + \rho, \epsilon_k - \delta_1) = 0 = (\Lambda, \epsilon_1 - \epsilon_k)$. In the former case $V(\Lambda)$ is typical, while in the latter case it is atypical.*

The so-called contravariant and covariant tensor irreps for $gl(m|n)$ were studied using the Young super-diagram method in [12, 1]. From [16], we have

Proposition 1 *The contravariant and covariant tensor irreps of $gl(m|n)$ are unitary irreps of type (1) and type (2), respectively.*

Except for the typical type (1) unitary irreps with $c = (\Lambda + \rho, \epsilon_m - \delta_n) > 0$ being *non-integral*, the rest of the type (1) unitary irreps include the contravariant tensor irreps. Similarly the atypical type (2) unitary irreps and the typicals with $c' = (\Lambda + \rho, \epsilon_1 - \delta_1)$ being an integer, include the covariant tensor irreps of $gl(m|n)$.

Consider a contravariant tensor irrep of $gl(m|n)$ which is characterized by a partition of the Young super-diagram,

$$P = (p_1, p_2, \dots, p_m, p_{m+1}, p_{m+2}, \dots, p_N) \quad (\text{B.3})$$

where the box numbers $p_a \in \mathbf{Z}^+$, $p_a \geq p_{a+1}$, $\forall a$ and $p_{m+1} \leq n$ is assumed. Associated with each partition P , there exists a unique highest weight

$$\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_m | \bar{\Lambda}_1, \bar{\Lambda}_2, \dots, \bar{\Lambda}_n) \quad (\text{B.4})$$

with

$$\Lambda_i = p_i, \quad i = 1, 2, \dots, m \quad (\text{B.5})$$

and the $\bar{\Lambda}_i$'s defined by

$$\sum_{i=1}^n \bar{\Lambda}_i \delta_i = \sum_{i_1=1}^{p_{m+1}} \delta_{i_1} + \sum_{i_2=1}^{p_{m+2}} \delta_{i_2} + \dots + \sum_{i_{N-m}=1}^{p_N} \delta_{i_{N-m}} \quad (\text{B.6})$$

When $p_m \geq n$, the irreducible $gl(m|n)$ -module associated with the partition P is typical and type (1) unitary. If $p_m < n$, it is atypical and type (1) unitary.

The reduction of tensor products in the non-graded case extendeds to the graded case [12, 1]: the Kronecker products for the same type of irreps of $gl(m|n)$ (i.e. contravariant with contravariant, or covariant with covariant, but not a mixture of them) are governed by the usual (Littlewood-Richardson) rule. However, there is a major difference: for $gl(k)$ the Young tableaux can have at most $k - 1$ rows but for $gl(m|n)$ the Young super-tableaux can have any number of rows. This is because for superalgebras “antisymmetrization” of basis states implies antisymmetrization of bosonic parts but symmetrization of fermionic parts and one can continue the “antisymmetrization” process up to infinity.

We should emphasize that when c (resp. c') is non-integral, there exists non-trivial one-parameter family of typical type (1) (resp. type (2)) unitary irreps, which are not tensorial. Thus those irreps cannot be dealt with by the Young super-diagram techniques. In particular, for any one-parameter family of Kac modules of $gl(m|n)$ with highest weights of the form $\Lambda(\alpha) \equiv \Lambda + \alpha \sum_{i=1}^m \epsilon_i$, $\pi_{\Lambda(\alpha)}$ is both unitary and typical for $|\alpha|$ sufficiently large. We then know that $V(\Lambda(\alpha)) \otimes V(\Lambda(\beta))$ is completely reducible. This provides an initial step in calculating the tensor product decomposition in previous sections.

As shown in [17, 28, 39, 40, 41, 29], all of the above considerations apply equally well to the type-I quantum superalgebra $U_q(sl(m|n))$. In particular, the classification theorems still hold provided that Λ is real and $q > 0$.

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